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Liberating the Subgradient Optimality Conditions from Constraint Qualifications

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Abstract. In convex optimization the significance of constraint qualifications is evidenced by the simple duality theory, and the elegant subgradient optimality conditions which completely characterize a minimizer. However, the constraint qualifications do not always hold even for finite dimensional optimization problems and frequently fail for infinite dimensional problems. In the present work we take a broader view of the subgradient optimality conditions by allowing them to depend on a sequence of ϵ-subgradients at a minimizer and then by letting them to hold in the limit. Liberating the optimality conditions in this way permits us to obtain a complete characterization of optimality without a constraint qualification. As an easy consequence of these results we obtain optimality conditions for conic convex optimization problems without a constraint qualification. We derive these conditions by applying a powerful combination of conjugate analysis and ϵ-subdifferential calculus. Numerical examples are discussed to illustrate the significance of the sequential conditions.

Key words: necessary and sufficient conditions, ϵ-subdifferentials, sequential optimality conditions, convex optimization, semidefinite programs.

1. Introduction

Consider the convex optimization model problem

$$(P) \quad \text{Minimize } f(x) \\ \text{subject to } x \in C, \quad -g(x) \in S,$$

where C is a closed convex subset of a reflexive Banach space X, S is a closed convex cone of another reflexive Banach space Z, which does not necessarily have non-empty interior, $f: X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous convex function and $g: X \to Z$ is an S-convex mapping, i.e., convex with respect to the cone S. In the case where g is continuous and f is continuous at a feasible point $a \in \text{dom}\ f$ of (P), a constraint qualification ensures that the point a is a minimizer of (P) if and only if there exists $\lambda \in S^+, u \in \partial f(a), v \in \partial(\lambda \circ g)(a)$ and $w \in N_C(a)$ satisfying $u + v + w = 0$ and $(\lambda \circ g)(a) = 0$. Here S^+ is the dual cone of S, $N_C(a)$ is the normal cone of C at a and $\partial f(a)$ and $\partial(\lambda \circ g)(a)$ are the convex subdifferentials of f and $\lambda \circ g$ at a, respectively. Unfortunately, the constraint qualifications do not always hold even for the finite dimensional optimization problems (P) and frequently fail for problems (P) in infinite dimensions, and they hinder applications and numerical solution methods. The geometric constraint, $x \in C$, has often been the main source of constraint qualification violation for problems (P), where the interior of C may be empty.

Over the years a great deal of attention has been focussed on developing subgradient optimality conditions that do not use constraint qualifications (see [1, 12, 16] and other references therein). As a result various modified subgradient optimality conditions have been given in the literature [1, 10, 16]. In the present work we take a general view of the subgradient optimality conditions by allowing them to depend on a sequence of ϵ-subgradients at a minimizer and then letting them to hold in the limit. Very recently, liberating the optimality conditions in this way permitted us to obtain a complete sequential characterization of optimality without a constraint qualification for the special case of the model problem (P), where f is a real-valued continuous convex function and $C = X$ (see [12]). However, even the sequential subgradient conditions, given in [12], may not be valid for the general model (P) as $\partial f(a)$ may be empty (see Example 3.1).

In this paper we show that a more general sequential subgradient optimality condition, that does not use constraint qualifications, always holds for the general case of (P). More precisely, we establish that a feasible point $a \in \text{dom}\ f$ is a minimizer of (P) if and only if there exist sequences $\{\varepsilon_n\} \subset \mathbb{R}_+$, $\{\lambda_n\} \subset S^+$, $\{u_n\}, \{v_n\}, \{w_n\} \subset X'$ such that $u_n \in \partial_{\varepsilon_n} f(a)$, $v_n \in \partial_{\varepsilon_n}(\lambda_n \circ g)(a)$, $w_n \in N_C^{\varepsilon_n}(a)$ and

$$u_n + v_n + w_n \to_* 0, \varepsilon_n \to 0 \quad \text{and} \quad (\lambda_n \circ g)(a) \to 0 \text{ as } n \to \infty.$$

where $\partial_\epsilon h(a)$ is the ϵ-subdifferential of a function h at a and the weak* convergence of a sequence $\{t_n\}$ of X' to t is denoted by $t_n \to_* t$. We derive the sequential condition by applying a powerful combination of conjugate analysis and ϵ-subdifferential calculus [4, 5, 7].

We also obtain another sequential condition involving only the subgradients at nearby points to a minimizer by an application of the Brondsted–Rockafellar theorem [2, 16] which paves the way for describing an ϵ-subgradient at a point in terms of the subgradients at nearby points. Using the theory of sequential subdifferential calculus [7], similar conditions have recently been given in [16] for the particular case of (P) in infinite dimensions, where S is assumed to be a normal cone and f is a real-valued convex function.

The paper is organized as follows. Section 2 explains some basic results on convex sets and functions, and brings out important dual connections between the feasible set and the epigraphs of conjugate functions that will be used later in the paper. Section 3 presents the sequential subgradient necessary and sufficient optimality conditions, and also provides complete characterizations of optimality for conic convex optimization problems. The significance of the sequential conditions is illustrated by numerical examples.

2. Preliminaries

We begin by fixing some definitions and notations. We assume throughout the paper that X and Z are reflexive Banach spaces. Let D be a closed convex set in Z. The continuous dual space of X will be denoted by X' and will be endowed with the weak* topology. For the set $D \subset X$, the *closure* of D will be denoted cl D. If a set $A \subset X'$, the expression cl A will stand for the weak* closure. The *indicator function* δ_D is defined as $\delta_D(x) = 0$ if $x \in D$ and $\delta_D(x) = +\infty$ if $x \notin D$. The *support function* [9] σ_D is defined by $\sigma_D(u) = \sup_{x \in D} u(x)$. The *normal cone* of D is given by $N_D(x) = \{v \in X' : v(y - x) \leqslant 0, \forall y \in D\}$ when $x \in D$, and $N_D(x) = \emptyset$; when $x \notin D$. Given $\varepsilon \geqslant 0$, the ε-*normal cone* of D is given by $N_D^{\varepsilon}(x) := \{v \in X' : \sigma_D(v) \leqslant v(x) + \varepsilon\} = \{v \in X' : v(y - x) \leqslant \varepsilon, \forall y \in D\}$ when $x \in D$, and $N_D^{\varepsilon}(x) = \emptyset$ when $x \notin D$. Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semi-continuous convex function. Then, the *conjugate* function of f, $f^* : X' \to \mathbb{R} \cup \{+\infty\}$, is defined by

$$f^*(v) = \sup\{v(x) - f(x) | x \in \text{dom } f\}$$

where the *domain* of f, dom f, is given by dom $f = \{x \in X | f(x) < +\infty\}$. The *epigraph of* f, epi f, is defined by

$$\text{epi } f = \{(x, r) \in X \times \mathbb{R} | x \in \text{dom } f, \ f(x) \leqslant r\}.$$

The *subdifferential* of f, $\partial f : X \rightrightarrows X'$ is defined as

$$\partial f(x) = \{v \in X' | f(y) \geqslant f(x) + v(y - x), \forall y \in X\},$$

and the ε-*subdifferential of* f, $\partial_\varepsilon f : X \rightrightarrows X'$ is defined as

$$\partial_\varepsilon f(x) = \{v \in X' | f(y) \geqslant f(x) + v(y - x) - \varepsilon, \forall y \in X\}.$$

It follows easily from the definitions of epi f^* of a proper convex function f and the ε-subdifferential of f that if $a \in \mathrm{dom}\, f$, then

$$\mathrm{epi}\ f^* = \bigcup_{\epsilon \geqslant 0} \{(v, v(a) + \epsilon - f(a)) | v \in \partial_\epsilon f(a)\}. \tag{2.1}$$

For details see [12]. Note also that for each $x \in X$, $\partial\delta_D(x) = N_D(x)$ and $\partial_\varepsilon \delta_D(x) = N_D^\varepsilon(x)$. It follows from the Fenchel–Moreau theorem if $f, g: X \to \mathbb{R} \cup \{+\infty\}$ are proper lower semi-continuous convex functions with $\mathrm{dom}\ f \cap \mathrm{dom}\ g \neq \emptyset$ then $\mathrm{epi}(f+g)^* = \mathrm{cl}(\mathrm{epi}\ f^* + \mathrm{epi}\ g^*)$. If, in addition, epi f^* + epi g^* is weak *closed then

$$\partial(f+g)(x) = \partial f(x) + \partial g(x), \quad \forall x \in \mathrm{dom}\ f \cap \mathrm{dom}\ g.$$

For details see [15, 3]. The mapping $g: X \to Z$ is S-convex if for every u, $v \in X$ and $t \in [0, 1]$, $g(tu + (1-t)v) - tg(u) - (1-t)g(v) \in -S$.

Let $A := C \cap g^{-1}(-S) = \{x \in C | -g(x) \in S\}$. The connection between the dual cone involving the feasible set of (P) and the epigraphs of conjugate functions involving the constraints is given by the following lemma. Note that, for convenience, we denote the composition of mappings by juxtaposition, i.e., $\lambda \circ g$ as λg.

LEMMA 2.1. *Let C be a closed convex subset of X and let $S \subset Z$ be a closed convex cone. Let $g: X \to Z$ be an S-convex function such that for each $\lambda \in S^+$, λg is lower semicontinuous. If $A \neq \emptyset$, then* $\mathrm{epi}\delta_A^* = \mathrm{cl}(\cup_{\lambda \in S^+} \mathrm{epi}\ (\lambda g)^* + \mathrm{epi}\ \delta_C^*)$.

Proof. Note that, for each $x \in X$, $\delta_A(x) = \sup_{\lambda \in S^+}(\lambda g + \delta_C)(x)$. So, $\delta_A^* = [\inf_{\lambda \in S^+}(\lambda g + \delta_C)^*]^{**}$. Since $\inf_{\lambda \in S^+}(\lambda g + \delta_C)^*$ is a convex function, it follows that

$$\mathrm{epi}\ \delta_A^* = \mathrm{cl}\left[\mathrm{epi}\ \left(\inf_{\lambda \in S^+}(\lambda g + \delta_C)^*\right)\right] = \mathrm{cl}(\cup_{\lambda \in S^+} \mathrm{epi}(\lambda g)^* + \mathrm{epi}\delta_C^*),$$

as $\mathrm{epi}(\lambda g + \delta_C)^* = \mathrm{cl}(\mathrm{epi}(\lambda g)^* + \mathrm{epi}\delta_C^*)$. □

A proof of Lemma 2.1, using a separation theorem [8] was initially given in [13]. A special case of Lemma 2.1 was given in [12], where g is continuous and $C = X$.

LEMMA 2.2. *Let $f: X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Let C be a closed convex subset of X and let $S \subset Z$ be a closed convex cone. Let $g: X \to Z$ be an S-convex function such that for each $\lambda \in S^+$, λg is lower semicontinuous. If $A \neq \emptyset$, then* $\mathrm{cl}(\mathrm{epi}\ f^* + \mathrm{epi}\delta_A^*) = \mathrm{cl}\ (\mathrm{epi} f^* + \cup_{\lambda \in S^+} \mathrm{epi}(\lambda g)^* + \mathrm{epi}\delta_C^*)$.

Proof. By Lemma 2.1, we see that epi $\delta_A^* = \mathrm{cl}(\cup_{\lambda\in S^+}\mathrm{epi}(\lambda g)^* + \mathrm{epi}\delta_C^*)$. Thus, $\mathrm{cl}(\mathrm{epi} f^* + \mathrm{epi}\delta_A^*) = \mathrm{cl}(\mathrm{epi}\ f^* + cl(\cup_{\lambda\in S^+}\mathrm{epi}(\lambda g)^* + \mathrm{epi}\ \delta_C^*)) = \mathrm{cl}(\mathrm{epi} f^* + \cup_\lambda \mathrm{epi}(\lambda g)^* + \mathrm{epi}\delta_C^*)$. □

The following version of the Brondsted–Rockafellar theorem [16] will be useful in deriving a sequential optimality conditions solely in terms of the subdifferentials of the functions involved in (P).

PROPOSITION 2.1. (*Brondsted–Rockafellar Theorem [2, 16]*). *Let $f\colon X\to \mathbb{R}\cap\{+\infty\}$ be a proper lower semicontinuous convex function. Then for any real number $\epsilon>0$ and any $u\in\partial_\epsilon f(a)$ there exist $x_\epsilon\in X$ and $u_\epsilon\in\partial f(x_\epsilon)$ such that*

$$\|x_\epsilon - a\|\leqslant\sqrt{\epsilon},\ \|u_\epsilon - u\|\leqslant\sqrt{\epsilon}\ \text{ and }\ |f(x_\epsilon)-u_\epsilon(x_\epsilon - a) - f(a)|\leqslant 2\epsilon.$$

3. Sequential Subgradient Conditions

We begin by establishing a sequential condition, in terms of epigraphs of conjugate functions, characterizing a solution point of (P).

THEOREM 3.1. *For the problem (P), let $a\in A\cap\mathrm{dom}\ f$. Then the point a is a minimizer of (P) if and only if there exist sequences $\{(u_n,\alpha_n)\}$, $\{(v_n,\beta_n)\}$, $\{(w_n,\gamma_n)\}\subset X'\times\mathbb{R}$ and $\{\lambda_n\}\subset Z'$ such that $(u_n,\alpha_n)\in\mathrm{epi}\ f^*$, $(v_n,\beta_n)\in\mathrm{epi}\ (\lambda_n g)^*$, $(w_n,\gamma_n)\in\mathrm{epi}\,\delta_C^*$, $\lambda_n\in S^+$,*

$$u_n+v_n+w_n\to_* 0\quad \text{and}\quad f(a)+\alpha_n+\beta_n+\gamma_n\to 0\ \text{as}\ n\to\infty.$$

Proof. [$\Longrightarrow$] Assume that a is a minimizer of (P). Then, $0\in\partial(f+\delta_A)(a)$, and by the definitions of the subdifferential and the conjugate function of $f+\delta_A$, $(0,-f(a))\in\mathrm{epi}(f+\delta_A)^*$. Now, by Lemma 2.2, epi $(f+\delta_A)^* = \mathrm{cl}(\mathrm{epi}\ f^* + \mathrm{epi}\,\delta_A^*) = \mathrm{cl}\ (\mathrm{epi} f^* + \cup_{\lambda\in S^+}\mathrm{epi}(\lambda g)^* + \mathrm{epi}\,\delta_C^*)$. So, there exists a sequence $\{(\bar u_n,\bar c_n)\}\subset X'\times\mathbb{R}$ such that $(\bar u_n,\bar c_n)\in\mathrm{epi}\ f^* + \cup_{\lambda\in S^+}\mathrm{epi}(\lambda g)^* + \mathrm{epi}\ \delta_C^*$, $\bar u_n\to_* 0$ and $\bar c_n\to -f(a)$. Thus there exist sequences $\{(u_n,\alpha_n)\}$, $\{(v_n,\beta_n)\}$, $\{(w_n,\gamma_n)\}\subset X'\times\mathbb{R}$ and $\{\lambda_n\}\subset Z'$ such that $(u_n,\alpha_n)\in\mathrm{epi}\, f^*$, $(v_n,\beta_n)\in\mathrm{epi}\,(\lambda_n g)^*$, $(w_n,\gamma_n)\in\mathrm{epi}\,\delta_C^*$, $\lambda_n\in S^+$,

$$\bar u_n = u_n+v_n+w_n\to_* 0\quad\text{and}\quad \bar c_n=\alpha_n+\beta_n+\gamma_n\to -f(a)\quad \text{as}\ n\to\infty.$$

Conversely, assume that preceding conditions hold. Let $x\in A$. Then, by the definition of the epigraph of a conjugate function, $f(x)\geqslant u_n(x)-\alpha_n$, $0\geqslant v_n(x)-\beta_n$, $0\geqslant w_n(x)-\gamma_n$. Adding these three inequalities, we obtain that

$$f(x)\geqslant u_n(x)+v_n(x)+w_n(x)-(\alpha_n+\beta_n+\gamma_n).$$

Passing to the limit as $n\to\infty$, we get that $f(x)\geqslant f(a)$. As this inequality holds for each $x\in A$, the point a is a minimizer of (P). □

THEOREM 3.2. *For the problem* (P), *let* $a\in A\cap \operatorname{dom} f$. *Then the point* a *is a minimizer of* (P) *if and only if there exist sequences* $\{\varepsilon_n\}\subset\mathbb{R}_+, \{\lambda_n\}\subset S^+, \{u_n\}, \{v_n\}, \{w_n\}\subset X'$ *such that* $u_n\in\partial_{\varepsilon_n}f(a), v_n\in\partial_{\varepsilon_n}(\lambda_n g)(a), w_n\in N_C^{\varepsilon_n}(a)$ *and*

$$u_n+v_n+w_n\to_* 0 \quad \textit{and} \quad \varepsilon_n\to 0, (\lambda_n g)(a)\to 0 \quad \textit{as } n\to\infty.$$

Proof. [$\Longrightarrow$] Assume that a is a minimizer of (P). Then, by Theorem 3.1 there exist sequences $\{(u_n,\alpha_n)\}$, $\{(v_n,\beta_n)\}$, $\{(w_n,\gamma_n)\}\subset X'\times\mathbb{R}$ and $\{\lambda_n\}\subset Z'$ such that $(u_n,\alpha_n)\in \operatorname{epi} f^*, (v_n,\beta_n)\in \operatorname{epi}(\lambda_n g)^*, (w_n,\gamma_n)\in \operatorname{epi}\delta_C^*, \lambda_n\in S^+$,

$$u_n+v_n+w_n=\bar{u}_n\to_* 0 \quad \text{and} \quad \alpha_n+\beta_n+\gamma_n=\bar{c}_n\to -f(a) \quad \text{as } n\to\infty.$$

Since

$$\operatorname{epi} f^*=\bigcup_{\epsilon\geqslant 0}\{(u,u(a)+\epsilon-f(a))|u\in\partial_\epsilon f(a)\},$$

$$\operatorname{epi}(\lambda_n g)^*=\bigcup_{\eta\geqslant 0}\{(u,u(a)+\eta-(\lambda_n g)(a))|u\in\partial_\eta(\lambda_n g)(a)\}$$

and

$$\operatorname{epi}\delta_C^*=\bigcup_{\zeta\geqslant 0}\{(v,v(a)+\zeta)|v\in\partial_\zeta\delta_C(a)\},$$

there exist sequences $\{\varepsilon_n\}$, $\{\eta_n\}$, $\{\zeta_n\}\subset\mathbb{R}_+$ such that

$$\begin{aligned}
&u_n\in\partial_{\epsilon_n}f(a) \quad \text{and} \quad \alpha_n=u_n(a)+\epsilon_n-f(a)\\
&v_n\in\partial_{\eta_n}(\lambda_n g)(a) \quad \text{and} \quad \beta_n=v_n(a)+\eta_n-(\lambda_n g)(a)\\
&w_n\in\partial_{\zeta_n}\delta_C(a) \quad \text{and} \quad \gamma_n=w_n(a)+\zeta_n.
\end{aligned}$$

Then, $\alpha_n+\beta_n+\gamma_n=(u_n+v_n+w_n)(a)-f(a)-(\lambda_n g)(a)+(\epsilon_n+\eta_n+\zeta_n)\geqslant(u_n+v_n+w_n)(a)-f(a)+(\epsilon_n+\eta_n+\zeta_n)$. Now, passing to the limit as $n\to\infty$, we see that

$$-f(a)=\lim_{n\to\infty}(\alpha_n+\beta_n+\gamma_n)\geqslant -f(a)+\lim_{n\to\infty}(\epsilon_n+\eta_n+\zeta_n);$$

thus, $\lim_{n\to\infty}(\epsilon_n+\eta_n+\zeta_n)\leqslant 0$. This gives us that $\lim_{n\to\infty}\epsilon_n=0$, $\lim_{n\to\infty}\eta_n=0$ and $\zeta_n\to 0$ as $\{\epsilon_n\},\{\eta_n\}$ and $\{\zeta_n\}\subset\mathbb{R}_+$.

Furthermore, as $\alpha_n+\beta_n+\gamma_n=(u_n+v_n+w_n)(a)-f(a)-(\lambda_n g)(a)+(\epsilon_n+\eta_n+\zeta_n)$, we obtain that $\lim_{n\to\infty}(\lambda_n g)(a)=0$. Let $\varepsilon_n=\max\{\epsilon_n,\eta_n,\zeta_n\}$. Then $\varepsilon_n\to 0$ as $n\to+\infty$ and $u_n\in\partial_{\varepsilon_n}f(a), v_n\in\partial_{\varepsilon_n}(\lambda_n g)(a), w_n\in N_C^{\varepsilon_n}(a)$.

[$\Longleftarrow$]The proof of the converse implication is similar to the one in Theorem 3.1 and so is omitted. □

Remark 3.1. Another version of the proof of Theorem 3.2 can also be given by using the formula (see [7]),

$$\partial(f+\delta_A)(a)=\bigcap_{\epsilon>0}\mathrm{cl}[\partial_\epsilon f(a)+\partial_\epsilon\delta_A(a)],$$

and the subdifferential calculus of [6, 17], instead of using the link between $\partial(f+\delta A)(a)$ and $\mathrm{epi}(f+\delta_A)^*$, exploited in the above proof.

COROLLARY 3.1. *For the problem* (P), *let* $a\in A\cap\mathrm{dom}\ f$. *Then the point* a *is a minimizer of* (P) *if and only if there exist sequences* $\{\lambda_n\}\subset S^+,\{x_n\}\subset\mathrm{dom}\ f,\{y_n\}\subset\mathrm{dom}\ (\lambda_n g),\{z_n\}\subset C,\{u_n\},\{v_n\},\{w_n\}\subset X'$ *such that* $u_n\in\partial f(x_n), v_n\in\partial(\lambda_n g)(y_n), w_n\in N_C(z_n)$ *and*

$$\begin{aligned}&u_n+v_n+w_n\to_* 0, \|x_n-a\|\to 0, \|y_n-a\|\to 0, \|z_n-a\|\to 0,\\&f(x_n)-u_n(x_n-a)-f(a)\to 0, (\lambda_n g)(y_n)-v_n(y_n-a)\to 0, \quad \textit{and}\\&w_n(z_n-a)\to 0,\ \textit{as}\ n\to\infty.\end{aligned}$$

Proof. By Theorem 3.2, we see that if a is a minimizer of (P), then there exist sequences $\{\varepsilon_n\}\subset\mathbb{R}_+,\{\lambda_n\}\subset S^+,\{\bar u_n\},\{\bar v_n\},\{\bar w_n\}\subset X'$ such that $\bar u_n\in\partial_{\varepsilon_n}f(a),\bar v_n\in\partial_{\varepsilon_n}(\lambda_n g)(a),\bar w_n\in N_C^{\varepsilon_n}(a)$ and

$$\bar u_n+\bar v_n+\bar w_n\to_* 0\quad\text{and}\quad\varepsilon_n\to 0, (\lambda_n g)(a)\to 0\quad\text{as } n\to\infty.$$

Using Proposition 2.1, we can find sequences $\{x_n\}\subset\mathrm{dom}f,\{y_n\}\subset\mathrm{dom}\ (\lambda_n g)$, $\{z_n\}\subset C,\{u_n\},\{v_n\},\{w_n\}\subset X'$, $u_n\in\partial f(x_n), v_n\in\partial(\lambda_n g)(y_n)$ and $w_n\in N_C(z_n)$ such that

$$u_n+v_n+w_n\to_* 0, \|x_n-a\|\to 0, \|y_n-a\|\to 0, \|z_n-a\|\to 0\quad\text{as } n\to\infty$$

and

$$\begin{aligned}&f(x_n)-u_n(x_n-a)-f(a)\to 0, (\lambda_n g)(y_n)-v_n(y_n-a)\to 0,\\&w_n(z_n-a)\to 0,\quad\text{as } n\to\infty\end{aligned}$$

The proof of the converse implication follows along the same line of arguments as presented earlier in this section and so is omitted. □

THEOREM 3.3. *For the problem* (P)*, assume that* $f: X \to \mathbb{R}$ *is a continuous convex function. If* $a \in A$ *is a minimizer of problem* (P) *if and only if there exist* $u \in \partial f(a)$ *and sequences* $\{\varepsilon_n\}, \{\lambda_n\} \subset \mathbb{R}_+$ *and* $\{u_n\}$, $\{v_n\} \subset X'$ *such that* $u_n \in \partial_{\varepsilon_n}(\lambda_n g)(a)$, $v_n \in N_C^{\varepsilon_n}(a)$ *and*

$$u + u_n + v_n \to_* 0, \varepsilon_n \to 0, (\lambda_n g)(a) \to 0 \quad \text{as } n \to +\infty.$$

Proof. The point a is a minimizer of problem (P) if and only if $0 \in \partial(f + \delta A)(a) = \partial f(a) + \partial\delta_A(a)$. Thus, there exist $u \in \partial f(a)$ and $v \in \partial\delta_A(a)$ such that $u + v = 0$. By Lemma 2.1, we have that epi $\delta_A^* = \text{cl}(\cup_{\lambda \in S^+} \text{epi}(\lambda g)^* + \text{epi } \delta_C^*)$. Since $v \in \partial\delta_A(a)$, $(v, v(a)) \in \text{epi } \delta_A^*$, there exist sequences $\{(u_n, \alpha_n)\}, \{(v_n, \beta_n)\}, \{(w_n, \gamma_n)\} \subset X' \times \mathbb{R}$ and $\{\lambda_n\} \subset Z'$ such that $(u_n, \alpha_n) \in \text{epi } f^*$, $(v_n, \beta_n) \in \text{epi } (\lambda_n g)^*$, $(w_n, \gamma_n) \in \text{epi } \delta_C^*$, $\lambda_n \in S^+$,

$$u_n + v_n \to_* v \quad \text{and} \quad \beta_n + \gamma_n \to v(a) - \delta_A(a) = v(a) \text{ as } n \to \infty.$$

Now, as in the proof of Theorem 3.2, we obtain that $\lim_{n\to\infty}(\lambda_n g)(a) = 0$ and $u_n + v_n \to_* v$. Letting $\varepsilon_n = \max\{\eta_n, \zeta_n\}$, we see that $u_n \in \partial_{\varepsilon_n}(\lambda g)(a)$, $v_n \in \partial_{\varepsilon_n}\delta_C(a) = N_C^{\epsilon_n}(a)$, so, $u_n + v_n \to_* v = -u$ and $\varepsilon_n \to 0$, as $n \to +\infty$.

The following example illustrates that the sequential condition, given in [12] (see Theorem 3.3) does not hold, whereas the sequential condition of Theorem 3.2 holds.

EXAMPLE 3.1. Consider the following convex programming problem:

$$\begin{array}{lll} (P1) & \text{Minimize} & f(x) \\ & \text{subject to} & g(x) = \max\{0, x\} \leqslant 0 \\ & & x \in C, \end{array}$$

$$\text{where } f(x) = \begin{cases} -\sqrt{x} & if\, x \geqslant 0 \\ +\infty & \text{otherwise,} \end{cases} \quad \text{and } C = [-1, 1].$$

Clearly, f is a lower semicontinuous convex function, g is a continuous convex function and C is a closed convex set. The feasible set $A = [-1, 0]$ and $a = 0$ is the minimizer of problem (P1). Since $\partial f(a) = \emptyset$, for any $\lambda \in S^+$ and $\{\varepsilon_n\} \subset R_+$, $0 \notin \partial f(a) + \partial_{\varepsilon_n}(\lambda g)(a) + N_C^{\varepsilon_n}(a)$.

Let $\lambda_n = n$ and let $\varepsilon_n = \frac{1}{\sqrt{n}}$. Then, $-n \in \partial_{\varepsilon_n} f(a)$ and $n \in \partial(\lambda_n g)(a)$. Indeed, we can easily verify that for any $x \in X$, $-nx \leqslant f(x) + \varepsilon_n$, since

$$-nx \begin{cases} < +\infty = f(x) & \text{if } x<0 \\ \leqslant 0 \leqslant -\sqrt{x}+\varepsilon_n = f(x)+\varepsilon_n & \text{if } 0 \leqslant x \leqslant \frac{1}{n} \\ < -1 < -\sqrt{x} < f(x)+\varepsilon_n & \text{if } \frac{1}{n} < x < 1 \\ \leqslant -x \leqslant -\sqrt{x} < f(x)+\varepsilon_n & \text{if } x \geqslant 1 \end{cases}$$

and $nx \leqslant n\max\{0,x\}+\varepsilon_n = (\lambda_n g)(x)+\varepsilon_n$. Thus, $-n \in \partial_{\varepsilon_n} f(a)$ and $\partial_{\varepsilon_n}(\lambda_n g)(a) = (-\infty, n]$, $N_C^{\varepsilon_n}(a) = [-\varepsilon_n, \varepsilon_n]$. Let $u_n = -n$, $v_n = n$, $w_n = \varepsilon_n$. Then, $u_n + v_n + w_n = \varepsilon_n \to 0$ and $(\lambda_n g)(a) = 0$.

THEOREM 3.4. *For the problem (P), let $a \in A\cap$ dom f. Assume that the set* epi $f^* + \cup_{\lambda\in S}+$epi $(\lambda g)^*+$epi δ_C^* *is weak* closed. Then a is a minimizer of problem (P) if and only if there exists $\lambda \in S^+$ such that*

$$0 \in \partial f(a) + \partial(\lambda g)(a) + N_C(a) \quad \text{and} \quad (\lambda g)(a) = 0.$$

Proof. Clearly, a is a minimizer of problem (P) if and only if $0 \in \partial(f+\delta_A)(a)$, i.e., $(0, -f(a)) \in$ epi $(f+\delta_A)(a)$. If epi $f^* + \cup_{\lambda\in S}$ + epi $(\lambda g)^*$ + epi $\delta^*{}_C$ is weak* closed, then, by Lemma 2.2,

$$\text{epi}(f+\delta_A)(a) = \text{epi } f^* \cup_{\lambda\in S} + \text{epi}(\lambda g)^* + \text{epi } \delta^*{}_C.$$

Thus, if a is a minimizer of problem (P), then $(0, -f(a)) \in$ epi $f^* + \cup_{\lambda\in S^+}epi(\lambda g)^*$ + epi $\delta^*{}_C$. Thus, there exist $\lambda \in S^+$, $(u,\alpha) \in$ epi f^*, $(v,\beta) \in \cup_{\lambda\in S}+$epi $(\lambda g)^*$ and $(w,\gamma) \in$ epi $\delta^*{}_C$ such that

$$0 = u + v + w \quad \text{and} \quad \alpha + \beta + r = -f(a).$$

Now simple calculations using the definition of epigraph and conjugate function show that $u \in \partial f(a)$, $v \in \partial(\lambda g)(a)$ and $w \in \partial\delta_C(a)$. Thus, $0 \in \partial f(a) + \partial(\lambda g)(a) + \delta_C(a)$. Furthermore, $-f(a) = -f(a) - (\lambda g)(a) + \epsilon + \eta + \zeta = -f(a) - (\lambda g)(a)$, we have that $(\lambda g)(a) = 0$.

The proof of sufficiency of the subgradient condition for optimality is well know and so is omitted.

The conclusion of Theorem 3.4 was given in Theorem 4.1 in [3] under an additional assumption that the set $\cup_{\lambda\in S^+}$epi $(\lambda g)^*$ + epi $\delta^*{}_C$ is weak* closed. The following example illustrates that the sequential condition of Theorem 3.3 holds; whereas the condition of Theorem 3.4 does not hold. □

EXAMPLE 3.2. Consider the following convex programming problem:

$$\begin{aligned} (P2) \quad & \text{Minimize } f(x) = -x + y \\ & \text{subject to } g(x) = \max\{0, x\} + y^2 \leqslant 0 \\ & \qquad\qquad x \in C, \end{aligned}$$

where $C = [-1, 1] \times [-1, 1]$.

Let $X=\mathbb{R}^2$. Obviously, f and g are all continuous convex functions on X. The feasible set $A=[-1,0]\times\{0\}$. Clearly, $a=(0,0)$ is the minimizer of problem (P2). Moreover, $\partial f(a)=\{(-1,1)\}$, for any $\lambda\geqslant 0$, $\partial(\lambda g)(a)=[0,\lambda]\times\{0\}$ and $N_C(a)=\{(0,0)\}$. Thus, for any $(a,b)\in\partial f(a)+\partial(\lambda g)(a)+N_C(a)$, we have that b = 1. Thus $0\notin\partial f(a)+\partial(\lambda g)(a)+N_C(a)$. Therefore, the subgradient condition of Theorem 3.4 does not hold.

To verify the sequential condition of Theorem 3.3, let $\varepsilon_n=\frac{1}{n}$, $\lambda_n=n^2$. Then, $N_C^{\varepsilon_n}(a)=\{(u,v)\mid\mid u\mid+\mid v\mid\leqslant\varepsilon_n\}$ and $(1,-1)\in\partial_{\varepsilon_n}(\lambda_n g)(a)$. Indeed, $x\leqslant\lambda_n\max\{0,x\}$ and $-y\leqslant\lambda_n y^2+\varepsilon_n$ for any $(x,y)\in X$. Thus, $-x+y\leqslant(\lambda_n g)(x)+\varepsilon_n, i.e., (-1,1)\in\partial_{\varepsilon_n}(\lambda_n g)(a)$. Let $u_n=(-1,1)\in\partial_{\varepsilon_n}(\lambda_n g)(a)$ and $v_n=(0,\varepsilon_n)\in N_C^{\varepsilon_n}$. Then, $-(u_n+v_n)\to(1,-1)\in\partial f(a)$ and $\lambda_n g(a)=0$.

Consider the conic-convex optimization model problem:

$$\text{(CCP)}\quad \begin{array}{ll}\text{Minimize} & f(x)\\ \text{subject to} & x\in C,\ Ax=b,\end{array}$$

where C is a closed convex cone of X, $f:X\to\mathbb{R}\cup\{+\infty\}$ is a proper lower semicontinuous convex function and $A:X\to Z$ is a continuous linear operator, and $b\in Z$. Many classes of constrainted interpolation problems and approximation problems can be modelled as the conic-convex problems (CCP). Various constraint qualifications for the problems (CCP) have been given in the literature (see, [1], [14]). They usually require a Slater type condition such as the condition that for some $x_0\in\operatorname{int}C$, $Ax_0=b$. This condition often fails in applications. We deduce a complete characterization of optimality for (CCP) without a constraint qualification.

COROLLARY 3.2. *Let $f:X\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous convex function, C be a closed convex cone of X, $A:X\to Z$ be a continuous linear mapping; let $b\in Z$. Suppose $a\in C\cap A^{-1}(b)$. Then a is a minimizer of problem* (CCP) *if and only if there exists $\{\lambda n\}\subset Z'$, $\{\epsilon_n\}\subset R_+$ and $\{u_n\},\{w_n\}\subset X'$, such that $u_n\in\partial_{\epsilon_n}f(a)$, $w_n\in N_C^{\epsilon_n}(a)$ and*

$$u_n+w_n-\lambda_n A\to 0 \quad\text{and}\quad \varepsilon_n\to 0 \quad\text{as } n\to\infty.$$

Proof. Let $g(x)=Ax-b$ and let $S=\{0\}$. Then the conclusion follows from Theorem 3.2. □

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